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Existence of Integral Manifolds for Impulsive Differential Equations in a Banach Space

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A theorem of existence for $t \to \pm\infty$ of integral manifolds of impulsive equations is proved under the assumption that the spectrum of the linear part of these equations may contain points lying in a neighborhood of the imaginary axis.

1. INTRODUCTION

Impulsive differential equations constitute a useful mathematical apparatus for the investigation of evolutionary processes in science and technology. However, the mathematical theory of these equations has developed rather slowly. This is due to the presence of such phenomena as merging of solutions, dying of solutions, "beating," loss of the property of autonomy, etc.

The development of the mathematical theory of impulsive differential equations began with the work of Mil'man and Myshkis (1960, 1963), while the work of Bainov *et al.* (1988*a*,*b*; 1989) marked the beginning of the mathematical theory of the same equations in abstract spaces. The first monograph dedicated to this subject was by Samoilenko and Perestyuk (1987).

In the present paper a theorem of the existence of integral manifolds bounded for $t \to \pm\infty$ for impulsive equations is proved under the assumption that the spectrum of the linear part of these equations may contain points lying in a neighborhood of the imaginary axis. In Samoilenko and Perestyuk (1987) integral manifolds for $t \to \infty$ or $t \to -\infty$ of impulsive equations in finite-dimensional space were investigated under the strong assumption that the spectrum does not intersect the imaginary axis.

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2. STATEMENT OF THE PROBLEM

Let X be a complex Banach space with norm $\|\cdot\|$. Consider the impulsive equation

$$\frac{dx}{dt} = Ax + F(t, x)|_{t \neq t_n} \tag{1}$$

$$\Delta x|_{t=t_n} = Bx(t_n - 0) + I_n(x(t_n - 0)) \qquad (n \in \mathbb{Z}) \tag{2}$$

where $\mathbb{Z}$ is the set of all integers; $\Delta x|_{t=\tau} = x(\tau+0) - x(\tau-0)$; $F(t, x): (-\infty, \infty) \times X \to X$ is a function which is continuous with respect to t for $t \neq t_n$ $(n = 1, 2, 3, \ldots)$ and with respect to x, and at the points t_n $(n = 1, 2, 3, \ldots)$ it has discontinuities of the first kind and is continuous from the left; $A, B: X \to X$ are linear bounded operators; $I_n: X \to X$ $(n \in \mathbb{Z})$ are continuous impulse operators; t_n $(n \in \mathbb{Z})$ are fixed moments of impulse effect which satisfy the conditions

$$\cdots < t_{-2} < t_{-1} < t_1 < t_2 < \cdots, \qquad \lim_{n \to \pm\infty} t_n = \pm\infty \tag{3}$$

Definition 1. A solution of impulsive equation (1) with condition (2) is called a piecewise continuous function $x(t)$ with discontinuities of the first kind at the points $t = t_n$ $(n = 1, 2, 3, \ldots)$ which for $t \neq t_n$ $(n = 1, 2, 3, \ldots)$ satisfies equation (1) and for $t = t_n$ satisfies the condition of a "jump" (2).

We say that condition H is satisfied if the following conditions hold:

H1. Uniformly in $t \in (-\infty, \infty)$ there exists the limit

$$\lim_{T \to \infty} \frac{i(t, t+T)}{T} = p < \infty$$

where $i(a, b)$ is the number of the points t_n lying in the interval (a, b).

H2. The operators A and B commute with each other and the operator $I + B$ has a logarithm $L_n(I + B)$ (I is the identity operator in the space X).

Set

$$\Lambda = \Lambda(p, A, B) = A + pL_n(I + B) \tag{4}$$

Assume that the spectrum $\sigma(\Lambda)$ of the operator Λ admits an α decomposition, i.e., $\sigma(\Lambda)$ can be decomposed into two spectral sets

$$\sigma(\Lambda) = \sigma_1(\Lambda) \cup \sigma_2(\Lambda)$$

so that for $\alpha > 0$ the following inequalities are valid:

$$|\mathrm{Re}\, \lambda| > \alpha \qquad \text{for} \quad \lambda \in \sigma_2(\Lambda)$$

$$|\mathrm{Re}\, \lambda| < \alpha \qquad \text{for} \quad \lambda \in \sigma_1(\Lambda)$$

As usual (see, for instance, Daleckii and Krein, 1974) by P_1, P_2, X_1, and X_2 we shall denote the spectral projectors and the subspaces X_1 and X_2 invariant with respect to Λ corresponding to the spectral sets $\sigma_1(\Lambda)$ and $\sigma_2(\Lambda)$. Then

$$X = X_1 \oplus X_2, \qquad X_i = P_i X \quad (i = 1, 2)$$

Let $\Lambda_i = P_i \Lambda$ $(i = 1, 2)$ and

$$\begin{aligned} \lambda^*(\alpha) &= \inf\{|\mathrm{Re}\,\lambda|: \quad \lambda \in \sigma_2(\Lambda)\} \\ \lambda_*(\alpha) &= \sup\{|\mathrm{Re}\,\lambda|: \quad \lambda \in \sigma_1(\Lambda)\} \end{aligned} \tag{5}$$

Then (see, for instance, Daleckii and Krein, 1974) for any $\delta_1 \in (\lambda_*(\alpha), \alpha)$ there exists a number $N_1 > 0$ such that

$$\|e^{\Lambda_1(t-\tau)}\| \leq N_1\, e^{\delta_1|t-\tau|} \tag{6}$$

and for any $\delta_2 \in (\alpha, \lambda^*(\alpha))$ there exists a number N_2 such that

$$\begin{aligned} \|P_2^+\, e^{\Lambda_2(t-\tau)}\| &\leq N_2\, e^{-\delta_2|t-\tau|} \qquad (t < \tau) \\ \|P_2^-\, e^{\Lambda_2(t-\tau)}\| &\leq N_2\, e^{-\delta_2|t-\tau|} \qquad (\tau < t) \end{aligned} \tag{7}$$

where $P_2^{\pm}$ are the spectral projectors of the operator Λ_2 corresponding to the spectral sets $\sigma_{\pm}(\Lambda_2)$ [$\sigma_{\pm}(\Lambda_2)$ are the parts of the spectrum $\sigma(\Lambda_2) = \sigma_2(\Lambda)$ containing the spectral points of Λ_2 with positive real part and negative real part, respectively].

Definition 2. An integral curve or trajectory of the solution $x(t)$ of impulsive equation (1), (2) we shall call the set of points $(t, x(t))$ in the extended phase space $(-\infty, \infty) \times X$.

Definition 3. An integral manifold of impulsive equation (1), (2) we shall call the set $\tilde{M} \subset (-\infty, \infty) \times X$ which consists of the integral curves of this equation.

We shall investigate integral manifolds of impulsive equation (1), (2) which are described by equations of the form

$$x_2 = \varphi(t, x_1) \tag{8}$$

with values in X_2 and defined for $t \in (-\infty, \infty)$ and $x_1 \in X_1$.

Definition 4. The function $\varphi(t, x_1)$ defined in (8) is called a parametrization of the respective integral manifold.

By L we shall denote the linear space of all functions $\varphi(t, x_1): (-\infty, \infty) \times X_1 \to X_2$ which with respect to their first argument are continuous for $t \neq t_n$, at the points $t = t_n$ have discontinuities of the first kind and are continuous from the left, and with respect to their second argument satisfy the Lipschitz condition with a constant independent of t.

By L_η denote the subset of L consisting of all functions $\varphi(t, x_1)$ satisfying the condition

$$\|\varphi(t, x_1) - \varphi(t, x_2)\| \leq \eta \|x_1 - x_2\| (x_1, x_2 \in X_1) \tag{9}$$

where $\eta = \text{const} > 0$.

By $L(c, \eta)$ denote the set of functions of L_η for which

$$\|\varphi(t, x_1)\| \leq c (t \in (-\infty, \infty), x_1 \in X_1) \tag{10}$$

where $c, \eta > 0$ are constants.

In the present paper integral manifolds for impulsive equation (1), (2) with parametrizations of the set $L(c, \eta)$ for given $c > 0$ and $\eta > 0$ are investigated.

Definition 5. The measurable function $a(s) : (-\infty, \infty) \to B$, where B is an arbitrary Banach space with norm $\|\cdot\|_B$, is called integrably bounded (with constant $\tilde{A}$) if

$$\sup_{t>0} \frac{1}{\tau} \int_t^{t+\tau} \|a(s)\|_B \, ds \leq \tilde{A} < \infty$$

We shall say that the function $F(t, x)$: $((-\infty, \infty) \backslash \{t_n\}) \times X \to X$ belongs to the class $K(m, q, M, Q)$ if $F(t, x)$ is continuously continuable in each interval $[t_n, t_{n+1}]$ and for $x, y \in X$ and $t \in ((-\infty, \infty) \backslash \{t_n\})$ the following estimates are valid:

$$\|F(t, x)\| \leq m(t), \qquad \|F(t, x) - F(t, y)\| \leq q(t) \|x - y\|$$

where the functions $m(t) \geq 0$ and $q(t) \geq 0$ are integrally bounded with constants M and G, respectively.

We shall say that the couple $\{F(t, x), I_n(x)\}$ belongs to the class $\tilde{K}(m, q, M, Q)$ if $F(t, x) \in K(m, q, M, Q)$ and, moreover, the following estimates are valid:

$$\|I_n(x)\| \leq M, \qquad \|I_n(x) - I_n(y)\| \leq Q \|x - y\| \qquad (n \in \mathbb{Z}; \quad x, y \in X)$$

3. AUXILIARY LEMMAS

Lemma 1. Let the following conditions be fulfilled:

1. Condition H holds.

2. The spectrum $\sigma(\Lambda)$ of the operator $\Lambda = A + pL_n(I + B)$ does not intersect the imaginary axis.

3. The inequality $\sup_{n \in \mathbb{Z}} \|h_n\| < \infty$ ($h_n \in X$, $n = 1, 2, 3, \ldots$) holds.

Then, if the function $f(t)$ is continuous for $t \neq t_n$ and at the points $t = t_n$ it has discontinuities of the first kind and is continuous from the left and is integrally bounded, then the impulsive equation

$$\frac{dx}{dt} = Ax + f(t)|_{t \neq t_n} \tag{11}$$

$$x(t_n + 0) - x(t_n) = Bx(t_n) + h_n \tag{12}$$

has a unique bounded solution $x(t)$ which is given by the formula

$$x(t) = \int_{-\infty}^{\infty} G(t, \tau)\, d\tau + \sum_{i=-\infty}^{\infty} G(t, t_i) h_i \tag{13}$$

where the function $G(t, \tau)$ has the form

$$G(t, \tau) = \begin{cases} -P_- \, e^{\Lambda(t-\tau)}\, (I+B)^{[i(t,\tau) - p|t-\tau|]}, & t < \tau \\ P_+ \, e^{\Lambda(t-\tau)}\, (I+B)^{[i(t,\tau) - p|t-\tau|]}, & \tau < t \end{cases} \tag{14}$$

P_+ and P_- are the spectral projectors corresponding to the spectral sets $\sigma_+(\Lambda)$ and $\sigma_-(\Lambda)[\sigma_\pm(\Lambda)$ are the parts of the spectrum $\sigma(\Lambda)$ of the operator Λ lying respectively in the right and left half-planes].

The proof of Lemma 1 is a minor modification of the proof of Theorem 1 (Bainov *et al.*, 1988*b*).

Remark 1. Under the assumptions made in Lemma 1 there exist constants $K > 0$ and $\delta > 0$ for which the following estimate is valid:

$$\|G(t, \tau)\| \leq K e^{-\delta|t-\tau|}, \qquad t, \tau \in (-\infty, \infty) \tag{15}$$

Definition 6. The function $G(t, \tau)$ is called the Green's function of the homogeneous impulsive equation

$$\frac{dx}{dt} = Ax|_{t \neq t_n}, \qquad \Delta x|_{t=t_n} = Bx(t_n)$$

Lemma 2. Let the conditions of Lemma 1 hold. Then for any $\rho > 0$ there exist constants M and Q such that if the couple $\{F(t, x), I_n(x)\}$ belongs to the class $\tilde{K}(m, q, M, Q)$, then the impulsive equation (1), (2) has a unique solution $x(t)$ lying in the ball S_ρ, i.e.,

$$\sup_{-\infty < t < \infty} \|x(t)\| \leq \rho$$

Proof. Let $x(t)$ be a solution of impulsive equation (1), (2) lying in the ball S_ρ. Then the function $F(t, x(t))$ is integrally bounded and by Lemma 1 the solution $x(t)$ of impulsive equation (1), (2) satisfies the equation

$$x(t) = \int_{-\infty}^{\infty} G(t, \tau) F(\tau, x(\tau))\, d\tau + \sum_{i=-\infty}^{\infty} G(t, t_i) I_i(x(t_i)) \tag{16}$$

Later the proof is carried out by means of the principle of the contracting mappings of Banach-Caccioppoli, proving in a standard way that if M and Q are small enough, then equation (16) has a unique solution which lies in the ball S_ρ.

Let the operator Λ defined in (4) admit an α decomposition and let P_1, P_2, X_1, and X_2 be the projectors and the corresponding subspaces of X invariant with respect to Λ defined in Section 2.

Then, if the integral manifold (8) contains the integral curve $(t, x(t))$ and if we set $\psi(t) = P_1 x(t)$, then the following equality is valid:

$$x(t) = \psi(t) + \varphi(t, \psi(t)) \tag{17}$$

Impulsive equation (1), (2) can be represented as a system of two impulsive equations

$$\begin{aligned} \frac{dx_1}{dt} &= A_1 x_1 + F_1(t, x_1 + x_2)\big|_{t \neq t_n} \\ \Delta x\big|_{t=t_n} &= B_1 x_1(t_n) + I_n^1(x_1(t_n) + x_2(t_n)) \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{dx_2}{dt} &= A_2 x_2 + F_2(t, x_1 + x_2)\big|_{t \neq t_n} \\ \Delta x_2\big|_{t=t_n} &= B_2 x_2(t_n) + I_n^2(x_1(t_n) + x_2(t_n)) \end{aligned} \tag{19}$$

where $x_i = P_i x$, $F_i = P_i F$, $B_i = P_i B$, $A_i = P_i A$, $I_n^i = P_i I_n$ $(i = 1, 2)$.

Let $(t, x(t))$ be an integral curve belonging to an integral manifold with parametrization $\varphi(t, x_1)$ of the set $L(c, \eta)$. Then the functions $x_1 = \psi(t) = P_1 x(t)$ and $x_2 = \varphi(t, \psi(t))$ satisfy the system of impulsive equations (18), (19). ■

Lemma 3 (Daleckii and Krein, 1974). Let the function $f: (-\infty, \infty) \times X \to X$ be continuous and satisfy the conditions

$$\|f(t, x)\| \leq M_1(t) + M_0(t)\|x\|$$

$$\|f(t, x) - f(t, y)\| \leq M_2(t)\|x - y\|$$

where $x, y \in X$, $t \in (-\infty, \infty)$, and the functions $M_0(t)$, $M_1(t)$, and $M_2(t)$ are integrally bounded.

Then for $x_0 \in X$ and $t \in (-\infty, \infty)$ the differential equation

$$\frac{dx}{dt} = f(t, x)$$

has a unique solution $x(t)$ $[t \in (-\infty, \infty)]$ satisfying the initial condition $x(0) = x_0$.

Lemma 4. Let the following conditions be fulfilled:

1. Condition H holds.
2. The operator $\Lambda = A + pL_n(I + B)$ admits an α decomposition.
3. $\varphi(t, x_1) \in L(c, \eta), \{F(t, x), I_n(x)\} \in \tilde{K}(m, q, M, Q)$.

Then the impulsive equation

$$\frac{d\psi}{dt} = A_1\psi + F_1(t, \psi + \varphi(t, \psi))|_{t \neq t_n} \tag{20}$$

$$\Delta\psi|_{t=t_n} = B_1\psi(t_n) + I_n^1(\psi + \varphi(t, \psi))|_{t=t_n} \tag{21}$$

has for any $x_{10} \in X_1$ fixed a unique solution $\psi(t) = \Psi(t, \tau, x_{10}|\varphi)$ for which

$$\Psi(\tau, \tau, x_{10}|\varphi) = x_{10}$$

Proof. Each integral curve of impulsive equation (20), (21) consists of pieces of integral curves of the ordinary equation (20).

The proof of Lemma 4 follows from the estimates

$$\|F_1(t, \psi' + \varphi(t, \psi')) - F_1(t, \psi'' + \varphi(t, \psi''))\| \leq q(t)(1 + \eta)\|\psi' - \psi''\| \tag{22}$$

$$\|I_n^1(\psi' + \varphi(t, \psi')) - I_n^1(\psi'' + \varphi(t, \psi''))|_{t=t_n}\| \leq Q(1 + \eta)\|\psi' - \psi''\| \tag{23}$$

and from Lemma 3.

Thus, Lemma 4 is proved. ■

We note that estimates (22) and (23) are valid for the operators F_2 and I_n^2 as well.

The function $\psi(t)$ can be represented in the form

$$\psi(t) = \phi_1(t, \tau)x_{10} + \int_\tau^t \phi_1(t, s)F_1(s, \psi(s) + \varphi(s, \psi(s)))\, ds$$
$$+ \sum_{\tau < t_i < t} \phi_1(t, t_i)I_i^1(\psi(t_i) + \varphi(t, \psi(t_i))) \tag{24}$$

where the operator-valued function

$$\phi_1(t, s) = e^{\Lambda_1(t-s)}\,(I + B_1)^{-p(t-s)+(t,s)} \qquad (t \geq s) \tag{25}$$

is the Cauchy evolutionary operator of the homogeneous impulsive equation

$$\frac{d\psi}{dt} = A_1\psi|_{t \neq t_n} \tag{26}$$

$$\Delta\psi|_{t=t_n} = B_1\psi(t_n) \tag{27}$$

Denote by $G_2(t, \tau)$ the Green's function of the homogeneous impulsive equation

$$\frac{dx_2}{dt} = A_2x_2|_{t \neq t_n} \tag{28}$$

$$\Delta x_2|_{t=t_n} = B_2x_2(t_n) \tag{29}$$

Lemma 5. Let the following conditions be fulfilled:

1. Condition H holds.
2. The operator $\Lambda = A + pL_n(I+B)$ admits an α decomposition.
3. $\varphi(t, x_1) \in L(c, \eta)$, $\{F(t, x), I_n(x)\} \in \tilde{K}(m, q, M, Q)$.

Then for sufficiently small values of M and Q the function $x_2(t) = \varphi(t, \psi(t))$, where $\psi(t)$ is a solution of the impulsive equation (20), (21), satisfies the integral equation

$$x_2(\tau) = \int_{-\infty}^{\infty} G_2(t, \tau) F_2(t, \psi(t) + x_2(t))\, dt + \sum_{i=-\infty}^{\infty} G_2(\tau, t_i) I_i^2(\psi(t_i) + x_2(t_i)) \tag{30}$$

The proof of Lemma 5 follows from the fact that the spectrum of the operator $\Lambda_2 = P_2\Lambda$ does not intersect the imaginary axis and from Lemma 2.

Let $x_1 \in X_1$ be a fixed element. For fixed τ choose such a solution $\psi(t)$ of the impulsive equation (20), (21) which is equal to x_1 for $t = \tau$:

$$\psi(t) = \Psi(t, \tau, x_1 | \varphi)$$

Then from (30) for the function $\varphi(t, x_1)$ the following integrodifferential equation is obtained:

$$\varphi(\tau, x_1) = \int_{-\infty}^{\infty} G_2(\tau, t) F_2(t, \Psi(t, \tau, x_1|\varphi) + \varphi(t, \Psi(t, \tau, x_1|\varphi))\, dt + \sum_{i=-\infty}^{\infty} G_2(\tau, t_i) I_i^2(\Psi(t_i, \tau, x_1|\varphi) + \varphi(t_i, \Psi(t_i, \tau, x_1(\varphi))) \tag{31}$$

The function $\varphi(t, x_1) \in L(c, \eta)$ which defines the integral manifold of the impulsive equation (1), (2) satisfies equation (31). Conversely, if equation (31) has a solution which satisfies conditions (9), (10), then this solution defines an integral manifold of the impulsive equation (1), (2) with a parametrization

$$\varphi(t, x_1) \in L(c, \eta)$$

For further investigations the following lemma is necessary.

Lemma 6. Let the following inequality be valid:

$$u(t) \le \int_{t_0}^{t} v(s) u(s)\, ds + F(t) + \sum_{t_0 < \tau_i < t} \beta_i u(\tau_i) + \sum_{t_0 < \tau_i < t} \alpha_i(t)$$

where the function $u(t)$ is piecewise continuous with discontinuities of the first kind at the points $t = \tau_i$ ($\tau_0 < \tau_1 < \cdots$, $\lim_{n\to\infty} \tau_n = \infty$), $v(s) \ge 0$ is a locally integrable function, $F(t)$ and $\alpha_i(t)$ are nondecreasing functions for $t\in[t_0,\infty)$, and $\alpha_i(t)$, $\beta_i\geq 0$.

Then the following estimate is valid:

$$u(t)\leq\left[F(t)+\sum_{t_0<\tau_i<t}\alpha_i(t)\right]\prod_{t_0<\tau_i<t}(1+\beta_i)\exp\left[\int_{t_0}^{t}v(s)\,ds\right]$$

Lemma 6 is a trivial consequence of the lemma of Gronwall–Bellman.

We shall estimate the norm of the Cauchy evolutionary operator ϕ_1 [see formula (25)]. Let condition H1 hold. Then for $\varepsilon>0$ the following estimate is valid:

$$\|(I+B_1)^{-p(t-s)+i(t,s)}\|\leq d(\varepsilon)\,e^{\varepsilon\|L_n(I+B_1)\|\cdot|t-s|}\qquad(t\geq s)\tag{32}$$

where $d(\varepsilon)>0$ is a constant.

From (6), (25), and (32) there follows the estimate

$$\|\phi_1(t,s)\|\leq N_1(\varepsilon)\,e^{\delta_1(\varepsilon)|t-s|}\tag{33}$$

where

$$\delta_1(\varepsilon)=\delta_1+\varepsilon\|L_n(I+B_1)\|\tag{34}$$

$$N_1(\varepsilon)=N_1 d(\varepsilon)\tag{35}$$

Lemma 7. Let the conditions of Lemma 5 hold and let the numbers $\delta_1(\varepsilon)>0$ and $N_1(\varepsilon)>0$ be defined by (34) and (35). Then the operator-valued function $\Psi(t,\tau,x_1|\varphi)$ which is a solution of the impulsive equation (20), (21) with initial condition $\Psi(\tau,\tau,x_1|\varphi)=x_1$ satisfies the inequality

$$\begin{aligned}&\|\Psi(t,\tau,x^{(1)}|\varphi^{(1)})-\Psi(t,\tau,x_1^{(2)}|\varphi^{(2)})\|\\&\quad\leq N_1(\varepsilon)\Bigg\{\|x_1^{(1)}-x_1^{(2)}\|\,e^{[\delta_1(\varepsilon)+\beta Q]|t-\tau|}[1+QN_1(\varepsilon)]^{i(\tau,t)}\\&\qquad+\Bigg[\left|\int_{\tau}^{t}q(s)\,e^{[\delta_1(\varepsilon)+\beta_Q](t-s)}\,e^{\beta_Q(s-\tau)}\sup_{x_1}\|\varphi^{(1)}(s,x_1)-\varphi^{(2)}(s,x_2)\|\,ds\right|\\&\qquad+\sum_{\tau<t_i<t}Q\,e^{\delta_1(\varepsilon)|t-t_i|}\,e^{\beta_Q|t-\tau|}\sup_{x_1}\|\varphi^{(1)}(t_i,x_1)-\varphi^{(2)}(t_i,x_2)\|\Bigg]\\&\qquad\times[1+QN_1(\varepsilon)]^{i(\tau,t)}\Bigg\}\end{aligned}\tag{36}$$

where $\beta_Q=N_1(\varepsilon)(1+\eta)Q\to 0$ as $Q\to 0$ and ε fixed.

Proof. Introduce the following notations:

$$x_1^{(1)}(t)=\Psi(t,\tau,x_1^{(1)}|\varphi^{(1)})$$

$$x_1^{(2)}(t)=\Psi(t,\tau,x_1^{(2)}|\varphi^{(2)})$$

$$x_1(t)=\Psi(t,\tau,x_1^{(1)}|\varphi^{(2)})$$

Then from formula (24) it follows that the functions $x_1^{(1)}(t)$, $x_1^{(2)}(t)$, and $x_1(t)$ satisfy, respectively, the equations

$$x(t) = \phi_1(t, \tau)x_1^{(1)} + \int_\tau^t \phi_1(t, s)F_1(s, x(s) + \varphi^{(1)}(s, x(s)))\, ds + \sum_{\tau < t_i < t} \phi_1(t_i, t)I_i^1(x(t_i) + \varphi^{(1)}(t_i, x(t_i)))$$

$$x(t) = \phi_1(t, \tau)x_1^{(2)} + \int_\tau^t \phi_1(t, s)F_1(s, x(s) + \varphi^{(2)}(s, x(s)))\, ds + \sum_{\tau < t_i < t} \phi_1(t_i, t)I_i^1(x(t_i) + \varphi^{(2)}(t_i, x(t_i)))$$

$$x(t) = \phi_1(t, \tau)x_1^{(1)} + \int_\tau^t \phi_1(t, s)F_1(s, x(s) + \varphi^{(2)}(s, x(s)))\, ds + \sum_{\tau < t_i < t} \phi_1(t_i, t)I_i^1(x(t_i)) + \varphi^{(2)}(t_i, x(t_i)))$$

Let $t \geq \tau$. In view of (33), (22), and (23) we obtain the inequalities

$$\begin{aligned}
&\|x_1^{(1)}(t) - x_1(t)\| \\
&\leq \Bigg\| \int_\tau^t \phi_1(t, s)[F_1(s, x_1^{(1)}(s) + \varphi^{(1)}(s, x_1^{(1)}(s))) \\
&\quad - F_1(s, x_1(s) + \varphi^{(1)}(s, x_1(s))) + F_1(s, x_1(s) + \varphi^{(1)}(s, x_1(s))) \\
&\quad - F_1(s, x_1(s) + \varphi^{(2)}(s, x_1(s)))]\, ds \Bigg\| \\
&\quad + \Bigg\| \sum_{\tau < t_i < t} \phi_1(t_i, t)[I_i^1(x_1^{(1)}(t_i) + \varphi^{(1)}(t_i, x_1^{(1)}(t_i))) \\
&\quad - I_i^1(x_1(t_i) + \varphi^{(1)}(t_i, x_1(t_i))) \\
&\quad + I_i^1(x_1(t_i) + \varphi^{(1)}(t_i, x_1(t_i))) - I_i^1(x_1(t_i) + \varphi^{(2)}(t_i, x_1(t_i)))] \Bigg\| \\
&\leq N_1(\varepsilon)(1 + \eta) \int_\tau^t q(s)\, e^{\delta_1(\varepsilon)(t-s)}\, \|x_1^{(1)}(s) - x_1(s)\|\, ds \\
&\quad + N_1(\varepsilon) \int_\tau^t q(s)\, e^{\delta_1(\varepsilon)(t-s)} \sup_{x_1} \|\varphi^{(1)}(s, x_1) - \varphi^{(2)}(s, x_1)\|\, ds \\
&\quad + \sum_{\tau < t_i < t} N_1(\varepsilon)(1 + \eta)Q\, e^{\delta_1(\varepsilon)(t-t_i)}\, \|x_1^{(1)}(t_i) - x_1(t_i)\| \\
&\quad + \sum_{\tau < t_i < t} N_1(\varepsilon)Q\, e^{\delta_1(\varepsilon)(t-t_i)} \sup_{x_1} \|\varphi^{(1)}(t_i, x_1) - \varphi^{(2)}(t_i, x_1)\|
\end{aligned}$$

Set $u(t)=e^{-\delta_1(\varepsilon)t}\|x_1^{(1)}(t)-x_1(t)\|$. Then for the function $u(t)$ we obtain the inequality

$$u(t)\le N_1(\varepsilon)(1+\eta)\int_\tau^t q(s)u(s)\,ds+N_1(\varepsilon)\int_\tau^t q(s)\,e^{-\delta_1(\varepsilon)s}$$
$$\times\sup_{x_1}\|\varphi^{(1)}(s,x_1)-\varphi^{(2)}(s,x_1)\|\,ds+\sum_{\tau<t_i<t}N_1(\varepsilon)(1+\eta)Qu(t_i)$$
$$+\sum_{\tau<t_i<t}N_1(\varepsilon)Q\,e^{-t_i\delta_1(\varepsilon)}\sup_{x_1}\|\varphi^{(1)}(t_i,x_1)-\varphi^{(2)}(t_i,x_1)\|$$

Lemma 6 implies the estimate

$$u(t)\le\Bigg[N_1(\varepsilon)\int_\tau^t q(s)\,e^{-\delta_1(\varepsilon)s}\sup_{x_1}\|\varphi^{(1)}(s,x_1)-\varphi^{(2)}(s,x_1)\|\,ds$$
$$+\sum_{\tau<t_i<t}N_1(\varepsilon)Q\,e^{-t_i\delta_1(\varepsilon)}\sup_{x_1}\|\varphi^{(1)}(t_i,x_1)-\varphi^{(2)}(t_i,x_1)\|\Bigg]$$
$$\times\prod_{\tau<t_i<t}[1+QN_1(\varepsilon)]\exp\left[N_1(\varepsilon)(1+\eta)\int_\tau^t q(s)\,ds\right]$$

hence

$$\|\Psi(t,\tau,x_1^{(1)}|\varphi^{(1)})-\Psi(t,\tau,x_1^{(1)}|\varphi^{(2)})\|$$
$$\le\Bigg\{N_1(\varepsilon)\int_\tau^t q(s)\,e^{\delta_1(\varepsilon)(t-s)}\sup_{x_1}\|\varphi^{(1)}(s,x_1)-\varphi^{(2)}(s,x_1)\|\,dx$$
$$+\sum_{\tau<t_i<t}N_1(\varepsilon)Q\,e^{\delta_1(\varepsilon)(t-t_i)}\sup_{x_1}\|\varphi^{(1)}(t_i,x_1)-\varphi^{(2)}(t_i,x_1)\|\Bigg\}$$
$$\times[1+QN_1(\varepsilon)]^{i(\tau,t)}\,e^{N_1(\varepsilon)(1+\eta)Q(t-\tau)}$$
$$\le N_1(\varepsilon)\Bigg\{\int_\tau^t q(s)\,e^{[\delta_1(\varepsilon)+\beta_Q](t-s)}\,e^{\beta_Q(s-\tau)}\sup_{x_1}\|\varphi^{(1)}(s,x_1)-\varphi^{(2)}(s,x_1)\|\,ds$$
$$+\sum_{\tau<t_i<t}N_1(\varepsilon)Q\,e^{\delta_1(\varepsilon)(t-t_i)}\,e^{\beta_Q(t-\tau)}\sup_{x_1}\|\varphi^{(1)}(t_i,x_1)-\varphi^{(2)}(t_i,x_1)\|\Bigg\}$$
$$\times[1+QN_1(\varepsilon)]^{i(\tau,t)}\tag{37}$$

where $\beta_Q=N_1(\varepsilon)(1+\eta)Q$.

The case $t<\tau$ is considered analogously.

For the expression $\|\Psi(t,\tau,x_1^{(1)}|\varphi^{(2)})-\Psi(t,\tau,x_1^{(2)}|\varphi^{(2)})\|$ for $t\ge\tau$ we obtain the estimate

$$\|\Psi(t,\tau,x_1^{(1)}|\varphi^{(2)})-\Psi(t,\tau,x_1^{(2)}|\varphi^{(2)})\|$$
$$\le N_1(\varepsilon)\,e^{\delta_1(\varepsilon)(t-\tau)}\,\|x_1^{(1)}-x_1^{(2)}\|[1+QN_1(\varepsilon)]^{i(\tau,t)}\,e^{N_1(\varepsilon)(1+\eta)Q(t-\tau)}$$
$$=N_1(\varepsilon)\|x_1^{(1)}-x_1^{(2)}\|\,e^{[\delta_1(\varepsilon)+\beta_Q](t-\tau)}\,[1+QN_1(\varepsilon)]^{i(\tau,t)}\tag{38}$$

The case $t<\tau$ is considered analogously.

Inequality (36) follows from inequalities (37) and (38).
Lemma 7 is proved. ■

4. MAIN RESULT

Theorem 1. Let the following conditions be fulfilled:
1. Condition H holds.
2. The operator $\Lambda = A + pL_n(I+B)$ admits an α decomposition.

Then for all numbers $c>0$ and $\eta>0$ there exist constants M_0 and Q_0 depending only on Λ, p, c, and η such that if the couple $\{F(t,x), I_n(x)\}$ belongs to the class $\tilde{K}(m,q,M,Q)$ for $0<Q<Q_0$, $0<M<M_0$, then the impulsive equation (1), (2) has an integral manifold with a parametrization of the set $L(c,\eta)$.

Proof. First we note that the conditions of Theorem 1 imply the validity of Lemmas 4, 5, and 7. We shall estimate the norm of the Green's function of the homogeneous equation (28), (29):

$$G_2(t,\tau) = \begin{cases} -P_2^- e^{\Lambda_2(t-\tau)} (I+B_2)^{[i(t,\tau)-p|t-\tau|]}, & t<\tau \\ P_2^+ e^{\Lambda_2(t-\tau)} (I+B_2)^{[i(t,\tau)-p|t-\tau|]}, & t>\tau \end{cases} \tag{39}$$

where the projectors P_2^+ and P_2^- were defined in the formulation of estimate (7).

Condition H1 and formula (39) imply the inequalities

$$\|(I+B_2)^{[i(t,\tau)-p|t-\tau|]}\| \le d_2(\varepsilon)\, e^{\varepsilon \|L_n(I+B_2)\| |t-\tau|} \tag{40}$$

$$\|G_2(t,\tau)\| \le N_2(\varepsilon)\, e^{-\delta_2(\varepsilon)|t-\tau|} \tag{41}$$

where $d_2(\varepsilon)>0$ is some constant,

$$\delta_2(\varepsilon) = \delta_2 - \varepsilon \|L_n(I+B_2)\| \tag{42}$$

$$N_2(\varepsilon) = N_2 d_2(\varepsilon) \tag{43}$$

From condition H1 it follows that for any $\varepsilon>0$ the estimate

$$i(t,\tau) \le c(\varepsilon) + (p+\varepsilon)|t-\tau| \tag{44}$$

is valid, where $c(\varepsilon)\ge 0$ is some constant.

Set

$$\delta_0(\varepsilon) = (p+\varepsilon)\ln[1+QN_1(\varepsilon)] \tag{45}$$

$$R_Q(\varepsilon) = [1+QN_1(\varepsilon)]^{c(\varepsilon)} \tag{46}$$

Then the following inequality holds:

$$[1+QN_1(\varepsilon)]^{i(t,\tau)} \leq R_Q(\varepsilon)\, e^{\delta_0(\varepsilon)|t-\tau|} \tag{47}$$

Set

$$\begin{aligned} \delta_Q(\varepsilon) &= \delta_2(\varepsilon) - \delta_1(\varepsilon) - \delta_0(\varepsilon) - \beta_Q \\ \beta_Q &= N_1(\varepsilon)(1+\eta)Q \end{aligned} \tag{48}$$

Then the following equality is valid:

$$\begin{aligned} \delta_Q(\varepsilon) = {} & [\delta_2 - \delta_1 - \varepsilon\|L_n(I+B_1)\| - \varepsilon\|L_n(I+B_2)\|] \\ & + (p+\varepsilon)\ln[1+QN_1(\varepsilon)] - N_1(\varepsilon)(1+\eta)Q \end{aligned} \tag{49}$$

By the choice of δ_1, δ_2 [see (5)-(7)] we have $\delta_1 < \delta_2$, hence we can choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{\delta_2 - \delta_1}{\|L_n(I+B_1)\| + \|L_n(I+B_2)\|} \tag{50}$$

From equality (49) it follows that there exists a number $Q_1 > 0$ depending on ε such that for all Q: $0 < Q < Q_1$ the following inequality holds:

$$\delta_Q(\varepsilon) > 0 \tag{51}$$

Consider the Banach space E consisting of all functions $\varphi(t, x_1) : (-\infty, \infty) \times X_1 \to X_2$ which are continuous for $t \neq t_n$, at $t = t_n$ have discontinuities of the first kind, are continuous from the left with respect to t, and are bounded on $(-\infty, \infty) \times X_1$ with a norm

$$|||\varphi||| = \sup_{t, x_1} \|\varphi(t, x_1)\| \tag{52}$$

The set $L(c, \eta)$ introduced in Section 2 is closed in E. We shall show that there exist constants $M_0 > 0$ and $Q_0 > 0$ such that for all M, Q: $0 < M < M_0$, $0 < Q < Q_0$, the operator S defined by the formula $S\varphi = \tilde{\varphi}$, where

$$\begin{aligned} \tilde{\varphi}(t, x_1) = {} & \int_{-\infty}^{\infty} G_2(\tau, t) F_2(t, \Psi(t, \tau, x_1|\varphi) + \varphi(t, \Psi(t, \tau, x_1|\varphi)))\, dt \\ & + \sum_{i=-\infty}^{\infty} G_2(\tau, t_i) I_i(\Psi(t_i, \tau, x_1|\varphi) + \varphi(t, \Psi(t, \tau, x_1|\varphi))) \end{aligned} \tag{53}$$

map the set $L(c, \eta)$ into itself, is contracting, and hence it has in it exactly one fixed point $\varphi(t, x_1)$.

First we shall show that for small values of M the function $\tilde{\varphi}$ satisfies condition (10), i.e., $|||\varphi|||\le c$. Indeed, estimate (41) implies the inequality

$$\|\tilde{\varphi}(t,\tau)\|\le N_2(\varepsilon)\left\{\int_{-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-\tau|}\, m(t)\, dt + M\sum_{i=-\infty}^{\infty} e^{-\delta(\varepsilon)|t-t_i|}\right\}$$

From condition H1 it follows that there exists such a number $l>0$ that

$$\left|\frac{i(t,t+l)}{l}-p\right|<1, \qquad t\in(-\infty,\infty) \tag{54}$$

i.e., any interval of length l contains not more than $(p+1)l$ points of the sequence $\{t_n\}$. Set

$$K(t)=\frac{2(p+1)l}{1-e^{-tl}}, \qquad H(t)=\frac{2e^t}{1-e^{-t}} \qquad (t>0) \tag{55}$$

Since $\delta_2(\varepsilon)>0$ [see (50)], then

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-t_i|}\\
&\quad\le \sum_{k=-\infty}^{\infty}\ \sum_{t+kl<t_i<t+(k+1)l} e^{-\delta_2(\varepsilon)|t-t_i|}\\
&\quad\le 2(p+1)l\left\{\sum_{j=0}^{\infty} e^{-j\delta_2(\varepsilon)l}\right\}=\frac{2(p+1)l}{1-e^{-\delta_2(\varepsilon)l}}=K[\delta_2(\varepsilon)]
\end{aligned} \tag{56}$$

Let $k_0<\tau\le k_0+1$. Then the following inequalities are valid:

$$\begin{aligned}
&\int_{-\infty}^{\tau} e^{\delta_2(\varepsilon)(t-\tau)}\, m(t)\, dt\\
&\quad\le \sum_{k=-\infty}^{k_0+1}\int_{k-1}^{k} e^{\delta_2(\varepsilon)(t-\tau)}\, m(t)\, dt\\
&\quad\le \sum_{k=-\infty}^{k_0+1} e^{\delta_2(\varepsilon)(k-\tau)}\int_{k-1}^{k} m(t)\, dt\le M e^{\delta_2(\varepsilon)(k_0+1-\tau)}\sum_{k=-\infty}^{0} e^{k\delta_2(\varepsilon)}\\
&\quad\le M\frac{e^{\delta_2(\varepsilon)}}{1-e^{-\delta_2(\varepsilon)}}=\frac{M}{2}H[\delta_2(\varepsilon)]
\end{aligned} \tag{57}$$

Analogously, the following inequality is proved:

$$\int_{\tau}^{\infty} e^{\delta_2(\varepsilon)(\tau-t)}\, m(t)\, dt\le\frac{M}{2}H[\delta_2(\varepsilon)]$$

From the above inequality and inequality (57) we obtain the inequality

$$\int_{-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-\tau|}\, dt\le MH[\delta_2(\varepsilon)] \tag{58}$$

From inequalities (53), (56) and (58), there follows the inequality

$$\|\tilde{\varphi}(t, \tau)\| \le N_2(\varepsilon)\{H[\delta_2(\varepsilon)] + K[\delta_2(\varepsilon)]\}M \tag{59}$$

Hence for all $M \in (0, M_0)$ where

$$M_0 = \frac{\varepsilon}{N_2(\varepsilon)\{H[\delta_2(\varepsilon)] + K[\delta_2(\varepsilon)]\}} \tag{60}$$

the function $\tilde{\varphi}(t, x_1)$ satisfies condition (10).

From (41), (53), (47), (51), (56), (58), and Lemma 7 there follow the estimates

$$\begin{aligned}
&\|\tilde{\varphi}(\tau, x_1^{(1)}) - \tilde{\varphi}(\tau, x_1^{(2)})\| \\
&\quad \le N_2(\varepsilon)(1+\eta)\left\{\int_{-\infty}^{\infty} q(t)\, e^{-\delta_2(\varepsilon)|t-\tau|} \right.\\
&\qquad \times \|\Psi(t, \tau, x_1^{(1)}|\varphi) - \Psi(t, \tau, x_1^{(2)}|\varphi)\|\, dt + \sum_{i=-\infty}^{\infty} Q\, e^{-\delta_2(\varepsilon)|t_i-\tau|} \\
&\qquad \left. \times \|\Psi(t_i, \tau, x_1^{(1)}|\varphi) - \Psi(t, \tau, x_1^{(2)}|\varphi)\| \right\} \\
&\quad \le N_2(\varepsilon)(1+\eta)\left\{\int_{-\infty}^{\infty} q(t)\, e^{-\delta_2(\varepsilon)|t-\tau|} N_1(\varepsilon)\|x_1^{(1)} - x_1^{(2)}\|\, e^{[\delta_1(\varepsilon)+\beta_Q]|t-\tau|} \right.\\
&\qquad \times [1 + QN_1(\varepsilon)]^{i(\tau,t)}\, dt + \sum_{i=-\infty}^{\infty} Q\, e^{-\delta_2(\varepsilon)|t_i-\tau|} N_1(\varepsilon)\|x_1^{(1)} - x_1^{(2)}\| \\
&\qquad \left. \times e^{[\delta_1(\varepsilon)+\beta_Q]|t_i-\tau|}[1 + QN_1(\varepsilon)]^{i(\varepsilon,t)} \right\} \\
&\quad \le N_2(\varepsilon)N_1(\varepsilon)(1+\eta)\|x_1^{(1)} - x_1^{(2)}\| \left\{\int_{-\infty}^{\infty} q(t)\, e^{[-\delta_2(\varepsilon)+\delta_1(\varepsilon)+\beta_Q]|t-\tau|} \right.\\
&\qquad \times R_Q(\varepsilon)\, e^{\delta_0(\varepsilon)|t-\tau|}\, dt + \sum_{i=-\infty}^{\infty} Q\, e^{[-\delta_2(\varepsilon)+\delta_1(\varepsilon)+\beta_Q]|t-\tau|} \\
&\qquad \left. \times R_Q(\varepsilon)\, e^{\delta_0(\varepsilon)|t_i-\tau|} \right\} \\
&\quad \le N_2(\varepsilon)N_1(\varepsilon)(1+\eta)\|x_1^{(1)} - x_1^{(2)}\| \left\{\int_{-\infty}^{\infty} q(t) \right.\\
&\qquad \left. \times e^{-\delta_Q(\varepsilon)|t-\tau|}\, dt + Q \sum_{i=-\infty}^{\infty} e^{-\delta_Q|t_i-\tau|} \right\} \\
&\quad \le N_2(\varepsilon)\beta_Q\{H[\delta_Q(\varepsilon)] + K[\delta_Q(\varepsilon)]\}\|x_1^{(1)} - x_1^{(2)}\|
\end{aligned} \tag{61}$$

Estimate (41) and Lemma 7 imply the estimates

$$\begin{aligned}
&\|\tilde{\varphi}_1(\tau, x_1) - \tilde{\varphi}_2(\tau, x_1)\| \\
&\quad \le N_2(\varepsilon) \int_{-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-\tau|} \| F_2(t, \Psi(t, \tau, x_1|\varphi_1) + \varphi_1(t, \Psi(t, \tau, x_1|\varphi_1))) \\
&\qquad - F_2(t, \Psi(t, \tau, x_1|\varphi_2) + \varphi_2(t, \Psi(t, \tau_1, x_1|\varphi_2)))\| \, dt \\
&\qquad + N_2(\varepsilon) \sum_{i=-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t_i-\tau|} \| I_i^2(\Psi(t_i, \tau, x_1|\varphi_1) \\
&\qquad + \varphi_1(t_i, \Psi(t_i, \tau, x_1|\varphi_1))) - I_i^2(\Psi(t_i, \tau, x_1|\varphi_2) \\
&\qquad + \varphi_2(t_i, \Psi(t_i, \tau, x_1|\varphi_1)))\| \\
&\quad \le N_2(\varepsilon) \int_{-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-\tau|} \\
&\qquad \times [W_1(t) + W_2(t)] q(t) \, dt + N_2(\varepsilon) \sum_{i=-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t_i-\tau|} \\
&\qquad \times [W_1(t_i) + W_2(t_i)] Q
\end{aligned} \tag{62}$$

where

$$W_1(t) = \|\Psi(t, \tau, x_1|\varphi_1) - \Psi(t, \tau, x_1|\varphi_2)\|$$
$$W_2(t) = \|\varphi_1(t, \Psi(t, \tau, x_1|\varphi_1)) - \varphi_2(t, \Psi(t, \tau, x_1|\varphi_2))\|$$

We shall consider the case $t \ge \tau$ (the case $t < \tau$ is considered analogously). From Lemma 7 for the function $W_1(t)$ we obtain the inequalities

$$\begin{aligned}
W_1(t) \le{} & N_1(\varepsilon) \int_{\tau}^{t} q(s) \, e^{\delta_1(\varepsilon)(t-s)} \, e^{\beta_Q|t-\tau|} \, ds \, |||\varphi_1 - \varphi_2||| \\
& \times R_Q(\varepsilon) \, e^{\delta_0(\varepsilon)|t-\tau|} + N_i(\varepsilon) \sum_{\tau<t_i<t} Q \, e^{\delta_1(\varepsilon)|t-t_i|} \\
& \times e^{\beta_Q|t-\tau|} R_Q(\varepsilon) \, e^{\delta_0(\varepsilon)|t-\tau|} |||\varphi_1 - \varphi_2||| \\
\le{} & N_1(\varepsilon) R_Q(\varepsilon) |||\varphi_1 - \varphi_2||| \, e^{[i\delta_0(\varepsilon)+\beta_Q]|t-\tau|} \\
& \times \left\{ \int_{\tau}^{t} q(s) \, e^{\delta_1(\varepsilon)(t-s)} \, ds + Q \sum_{\tau<t_i<t} e^{\delta_1(\varepsilon)|t-t_i|} \right\}
\end{aligned} \tag{63}$$

Set

$$L(z) = \frac{e^z}{e^z - 1} \quad (z > 0); \qquad P(z) = (p+1) l \frac{e^{lz}}{e^{lz} - 1} \quad (z > 0) \tag{64}$$

Let $t=\tau+k+w$ $(0\le w<1)$. Then

$$\int_\tau^t q(s)\,e^{\delta_1(\varepsilon)(t-s)}\,ds\le\int_\tau^{\tau+k+1} q(s)\,e^{\delta_1(\varepsilon)(t-s)}\,ds$$

$$\le\sum_{i=0}^{k}\int_{\tau+i}^{\tau+i+1} e^{\delta_1(\varepsilon)(t-\tau-i)}q(s)\,ds$$

$$\le\sum_{i=0}^{k} Q\,e^{\delta_1(\varepsilon)w\delta_1(\varepsilon)(k-i)}$$

$$\le Q\,e^{\delta_1(\varepsilon)w}\,\frac{e^{\delta_1(\varepsilon)(k+1)}}{e^{\delta_1(\varepsilon)}-1}=Q\frac{e^{\delta_1(\varepsilon)}}{e^{\delta_1(\varepsilon)}-1}\,e^{\delta_1(\varepsilon)(t-\tau)}$$

$$=QL[\delta_1(\varepsilon)]\,e^{\delta_1(\varepsilon)(t-\tau)} \tag{65}$$

Let $l>0$ be the number defined by (54) and let $t=\tau+ml+\theta$ $(0\le\theta<l)$. Then in view of (54) we obtain

$$\sum_{\tau<t_i<t} e^{\delta_1(\varepsilon)|t-t_i|}\le\sum_{j=0}^{m}\sum_{\tau+jl<t_i<\tau+(j+1)l} e^{\delta_1(\varepsilon)|t-t_i|}$$

$$\le\sum_{j=0}^{m}(p+1)l\,e^{\delta_1(\varepsilon)jl}=(p+1)l\frac{e^{\delta_1(\varepsilon)(m+1)l}-1}{e^{\delta_1(\varepsilon)l}-1}$$

$$\le(p+1)l\frac{e^{\delta_1(\varepsilon)l}}{e^{\delta_1(\varepsilon)l}-1}\,e^{\delta_1(\varepsilon)(ml+\theta)}=p[\delta_1(\varepsilon)]\,e^{\delta_1(\varepsilon)(t-\tau)} \tag{66}$$

From (63), (65), and (66) there follows the estimate

$$W_1(t)\le N_1(\varepsilon)R_Q(\varepsilon)|||\varphi_1-\varphi_2|||[L[\delta_1(\varepsilon)]+P[\delta_1(\varepsilon)]]\,e^{[\delta_1(\varepsilon)+\delta_0(\varepsilon)+\beta_Q]|t-\tau|} \tag{67}$$

For $W_2(t)$ the following estimate is valid:

$$W_2(t)\le|||\varphi_1-\varphi_2|||+\eta W_1 \tag{68}$$

Then from (62), (67), (56), and (68) we obtain

$$\|\tilde\varphi_1(\tau_1x_1)-\bar\varphi_2(\tau_1x_2)\|\le N_2(\varepsilon)\int_{-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-\tau|}q(t)$$

$$\times(W_1(t)+|||\varphi_1-\varphi_2|||+\eta W_1(t))+N_2(\varepsilon)Q\sum_{i=-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t_i-\tau|}$$

$$\times(W_1(t_i)+|||\varphi_1-\varphi_2|||+\eta W_1(t_i))\le|||\varphi_1-\varphi_2|||N_2(\varepsilon)$$

$$\times\left\{\left[\int_{-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t-\tau|}q(t)\,dt+Q\sum_{i=-\infty}^{\infty} e^{-\delta_2(\varepsilon)|t_i-\tau|}\right]\right.$$

$$+QN_1(\varepsilon)R_Q(\varepsilon)[L(\delta_1(\varepsilon)]+P[\delta_1(\varepsilon))]$$

$$\times\left[\left[\int_{-\infty}^{\infty} e^{-\delta_Q(\varepsilon)|t-\tau|}\,dt+\sum_{i=-\infty}^{\infty} e^{-\delta_Q|t_i-\tau|}\right]\right\}\le |||\varphi_1-\varphi_2|||Q$$

$$\le |||\varphi_1-\varphi_2|||Q\{N_2(\varepsilon)[H[\delta_2(\varepsilon)]+K[\delta_2(\varepsilon)]]+N_1(\varepsilon)R_Q(\varepsilon)$$

$$\times[L(\delta_1(\varepsilon))+P(\delta_1(\varepsilon))]R_Q(\varepsilon)[H[\delta_Q(\varepsilon)]+K[\delta_Q(\varepsilon)]]\} \qquad (69)$$

where the functions $L(z)$, $P(z)$, $H(t)$, and $K(t)$ are defined in (64) and (55) and continuous for $z>0$ and for $t>0$, respectively.

From equality (49) and inequality (50), it follows that

$$\lim_{Q\to 0}\delta_Q(\varepsilon)=\delta_2-\delta_1-\varepsilon\|L_n(I+B_1)\|-\varepsilon\|L_n(I+B_2)\|>0$$

$$\lim_{Q\to 0} R_Q(\varepsilon)=1$$

From (34), (50), and (51), there follow the inequalities $\delta_1(\varepsilon)>0$, $\delta_2(\varepsilon)>0$, and $\delta_Q(\varepsilon)>0$, whence in view of inequalities (59), (61), and (68), there follows the existence of such a number $Q_0>0$ that for $0<M<M_0$, $0<Q<Q_0$ the operator $\tilde{\varphi}=S\varphi$ is contracting in the set $L(c,\eta)$. From the theorem of Banach-Cacciopoli, it follows that this operator has a unique fixed point which is a solution of the equation $\varphi=S\varphi$. We shall stress that the choice of the numbers M_0 and Q_0 depends on the concrete choice of the number ε in inequality (50).

It remains to show that the integral manifold constructed contains each solution $x(t)$ of the impulsive equation (1), (2) for which for $t\in(-\infty,\infty)$ the following inequality is valid:

$$\|P_2x(t)\|\le c \qquad (70)$$

Let $x(t)$ be a solution of the impulsive equation (1), (2) which satisfies inequality (69) and let $\psi(t)=P_1x(t)$, $\chi(t)=P_2x(t)$. The function $(\psi(t),\chi(t))$ is a solution of the system of equations

$$\frac{d\psi}{dt}=A_1\psi+F_1(t,\psi(t)+\chi(t))|_{t\ne t_n}$$

$$\Delta\psi|_{t=t_n}=B_1\psi(t_n)+I_n^1(\psi(t_n)+\chi(t_n))$$

$$\chi(t)=\int_{-\infty}^{\infty} G_2(t,s)F_2(s,\psi(s,\psi(s)+\chi(s))\,ds \qquad (71)$$

$$+\sum_{i=-\infty}^{\infty} G_2(t_i,t)I_i(\psi(t_i)+\chi(t_i))$$

System (70) has a unique solution which satisfies the condition $\|\chi(t)\|\le c$ $[t\in(-\infty,\infty)]$. The proof of this fact is carried out by means of the techniques applied above.

Let $\psi_1(t)$ be a solution of equation (20) with initial condition $\psi_1(t_0)=\psi(t_0)$ and let $\chi_1(t)=\varphi(t, \psi_1(t))$ $(-\infty<t<\infty)$. Then the function $(\psi_1(t), \chi_1(t))$ also satisfies system (70). Hence $\psi(t)\equiv\psi_1(t)$ and $\chi(t)=\varphi(t, \psi(t))$ $(-\infty<t<\infty)$, i.e., the solution considered lies in the integral manifold $\tilde{M}$ with a parametrization $\varphi(t, x_1)$ of the set $L(c, \eta)$.

Theorem 1 is proved. ■

5. CONCLUSION

The results of the present paper can be extended in a standard way to impulsive equations with a linear operator A depending on t.

By means of techniques analagous to the one applied in this paper a theorem of the existence of integral manifolds bounded for $t\to+\infty$ or $t\to-\infty$ of the impulsive equation (1), (2) can be proved in the case when the spectrum of the linear operator of this equation has points lying in a neighborhood of the imaginary axis.

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